Last class:

found out that differentiating sine series for $x = f(x)$

$\longrightarrow$ got a non-converging cosine series

$\neq$ cosine series of $\frac{d}{dx}(x) = 1$

Clarification for this fact:

<u>Theorem</u>: $f: [0,L] \to \mathbb{R}$, $f'(x)$ piecewise smooth
(f continuous) ($\Rightarrow$ $f'(x)$ has Fourier series which converges)

ⓐ differentiating sine series for f term by term
$\Rightarrow$ get cosine series for f' <u>ONLY</u>
if $f(0) = f(L) = 0$

ⓑ differentiating cosine series for f term by term
$\Rightarrow$ get sine series for f'

Proof for a

given sine series of f

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x$$

differentiate sine series term by term:

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{B_n n\pi}{L} \cos \frac{n\pi}{L} x}$$

let $f'(x) \sim \sum_{n=0}^{\infty} A_n \cos \frac{n\pi}{L} x$

cosine series for f'

Question: Are these two series the same?

$$\Leftrightarrow \boxed{\begin{aligned} &A_0 = 0 \\ &A_n = \frac{B_n n\pi}{L} \qquad n > 0 \end{aligned}}$$

Calculate the A_n's !

$$A_0 = \frac{2}{L}\int_0^L f'(x)\,dx$$

$$= \frac{2}{L}\Big(f(L) - f(0)\Big) \overset{!}{=} 0$$

<u>$n>0$</u>

$$A_n = \frac{1}{L}\int_0^L f'(x)\cos\frac{n\pi}{L}x\,dx$$

int. by parts

$$= \frac{1}{L}\Big[f(x)\cos\frac{n\pi}{L}x\Big|_0^L + \int_0^L f(x)\Big(\cancel{-}\frac{n\pi}{L}\sin\frac{n\pi}{L}x\Big)dx$$

$$= \frac{1}{L}\Big[f(L)\underbrace{\cos n\pi}_{(-1)^n} - f(0) + \int_0^L f(x)\frac{n\pi}{L}\sin\frac{n\pi}{L}x\,dx\Big]$$

$$= \frac{1}{L}\left[(-1)^n f(L) - f(0)\right] + \frac{1}{L}\frac{n\pi}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

$$= \frac{1}{L}\left[(-1)^n f(L) - f(0)\right] + \frac{n\pi}{L} B_n$$

result: $\boxed{\text{Coefficient } A_n \text{ of cosine series of } f' = \frac{1}{L}\left[(-1)^n f(L) - f(0)\right] + \frac{n\pi}{L} \underbrace{B_n}_{\text{coeff of sine series for } f(x)}}$

$\Rightarrow \quad A_n = \frac{n\pi}{L} B_n \iff f(L) = 0 = f(0)$

($(-1)^n f(L) - f(0) = 0$ both for n odd and for n even !)

bad news: need to be careful if $f(0) \neq 0$ or $f(L) \neq 0$

good news: even if $f(0) \neq 0$ or $f(L) \neq 0$ we can calculate cosine series of f' from sine series of f via

here: f needs to be continuous on $[0, L]$!

Remark: Integrating Fourier series term by term is less complicated.

If F is an antiderivative of f

can obtain its Fourier series from the one of f

always

by integrating term by term

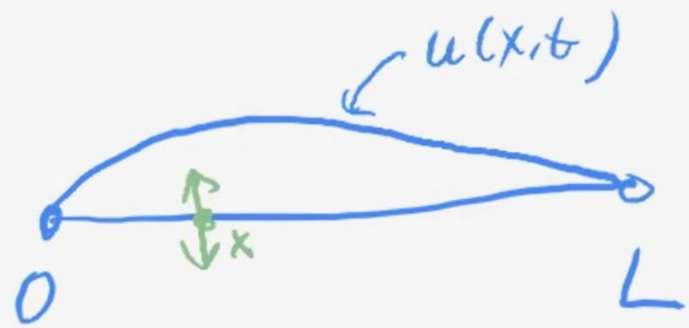
up to a constant.

4. Wave Equation

1-dim wave equation.

physical set-up.

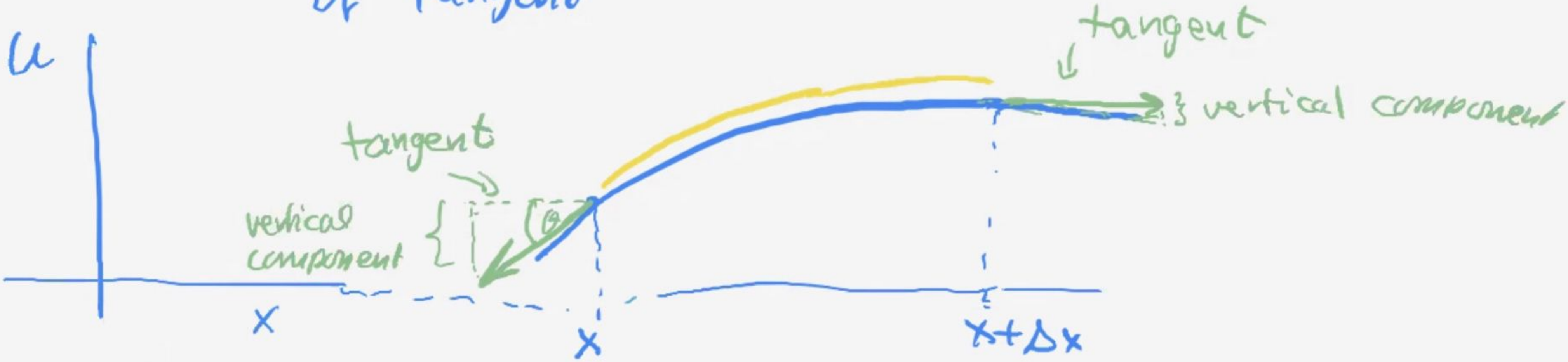
string spanned between



assumptions:

- string segments only move in vertical direction
- $u(x,t) = \underbrace{\ }_{\text{vertical}}$ position of string segm. at x at time t
- $\rho_0(x)$ mass density at x (usually constant)
- Newton's law: $\underset{\text{force}}{F} = \underset{\text{mass}}{m}\ \underset{\text{acceleration}}{a}$

- string perfectly flexible
  $\Rightarrow$ tension forces always going in direction of tangent



tension force pulls string segments towards ends of string

movement only in vertical direction $\Rightarrow$ only vertical component of tension force is relevant

relevant vertical part $\sin\theta \, T(x,t)$ ← tension force.

$\sin\theta \sim$ slope at x $\sim \frac{\partial u}{\partial x}(x,t)$

$\boxed{\text{relevant part of tension force} = T(x,t)\,\frac{\partial u}{\partial x}(x,t)}$

determine force on line segment between x and $x+\Delta x$

$$F = m\, a$$

$$= \underbrace{\rho_0(x)\,\Delta x}_{=m} \; \underbrace{\frac{\partial^2 u}{\partial t^2}}_{=a}$$

$$= \underbrace{\text{forces from tension}}_{\text{act on endpoints of line segments}} + \text{exterior forces (e.g. gravity)}$$

$$= \boxed{\frac{\partial u}{\partial x}(x+\Delta x,t)\,T(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\,T(x,t) + \underbrace{\rho_0(x)\,\Delta x}_{\text{mass}} \; \underbrace{Q(x,t)}_{\text{exterior force}}}$$

<u>get</u>

$$\rho_0(x)\,\Delta x \,\frac{\partial^2 u}{\partial t^2} =$$